UKMT Lockdown Problem

United Kingdom Mathematics Trust

1	2	3	4	5	6	7
30	31	32	33	34	35	36

Pair off the red and blue numbers, so that each blue number is divisible by the red number that it is paired with.

Now divide each blue number by the red number that it is paired with.

What is the sum of all the integers that you obtain in this way?

Day One

UKMT Lockdown Solution

84

The only way to pair off the red and blue numbers so each blue number is paired with a red number that it is divisible by is as shown below.

1	2	3	4	5	6	7
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
31	34	33	32	30	36	35

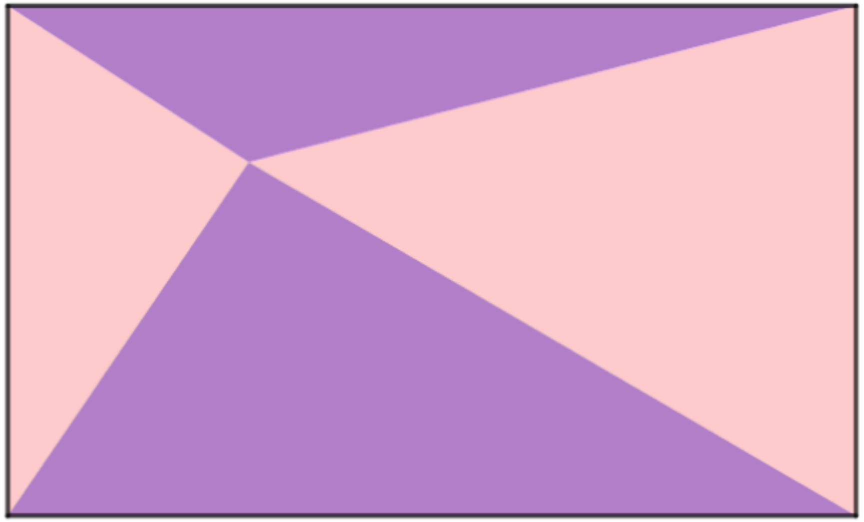
When the blue numbers are divided by the corresponding red numbers we obtain

$$\frac{31}{1}+\frac{34}{2}+\frac{33}{3}+\frac{32}{4}+\frac{30}{5}+\frac{36}{6}+\frac{35}{7}$$
$$=31+17+11+8+6+6+5=84.$$

United Kingdom Mathematics Trust

Day One

UKMT Lockdown Problem



The flag shown measures $100\,\text{cm} \times 60\,\text{cm}$.
It is divided into two purple triangles and two pink triangles.

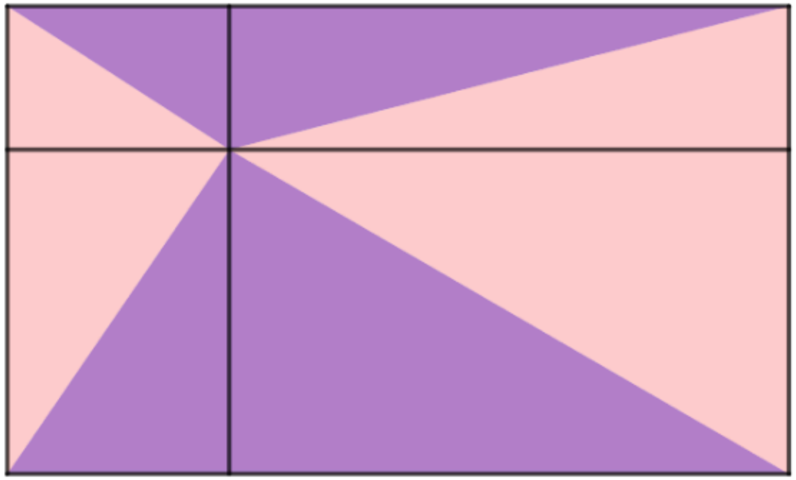
What is the total area of the two purple triangles?

United Kingdom Mathematics Trust

Day Two

UKMT Lockdown Solution

$3\,000\,\text{cm}^2$



The flag can be divided into four rectangles, as shown.

Half the area of each rectangle is purple.

Therefore half the area of the flag is purple.

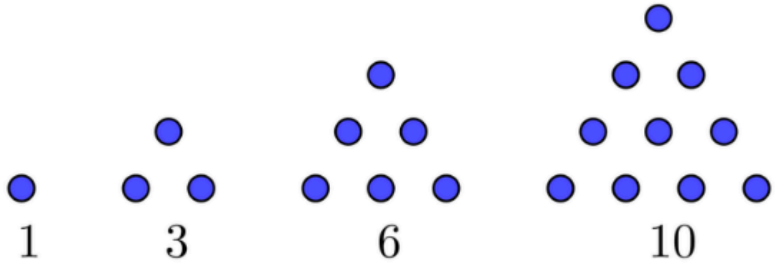
Hence the purple area is

$\frac{1}{2}(100 \times 60)\,\text{cm}^2 = \frac{1}{2}(6\,000)\,\text{cm}^2 = 3\,000\,\text{cm}^2$.

United Kingdom Mathematics Trust

Day Two

UKMT Lockdown Problem



A *triangular number* is the number of dots in a triangle of dots, such as those shown above. Thus the first four triangular numbers are $1, 3, 6$ and 10.

Note that 6 is twice 3.
Which is the next pair of triangular numbers, T and U where U is twice T? [Note that T and U are not consecutive triangular numbers. When you have found T and U you could to try find more of these pairs.]

UKMT
United Kingdom Mathematics Trust

Day Three

UKMT Lockdown Solution

105 and 210

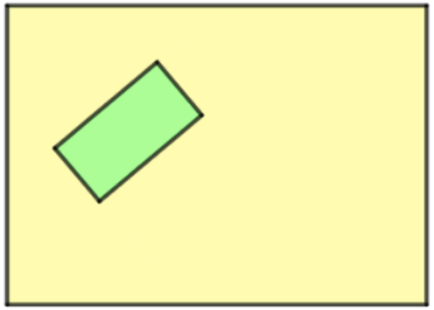
There are, in fact, infinitely many pairs, (T, U), of triangular numbers with $U = 2T$. This is not easy to prove.
Using the notation T_n for the nth triangular number, the first four of these pairs are:

$(T_2, T_3) = (3, 6)$; $(T_{14}, T_{20}) = (105, 210)$;
$(T_{84}, T_{119}) = (3570, 7140)$, and
$(T_{492}, T_{696}) = (121\,278, 242\,556)$.

United Kingdom Mathematics Trust

Day Three

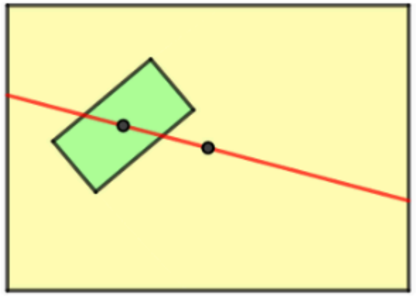
UKMT Lockdown Problem



The diagram shows one rectangle inside another rectangle.
Is it possible to draw one straight line which cuts each rectangle in half, that is, a straight line that cuts each rectangle into two quadrilaterals with equal areas?
If so, how many straight lines with this property is it possible to draw?

UKMT Lockdown Solution

Yes. There is just one line with the required property.



The straight lines that divide a rectangle in half are those that go through the centre of the rectangle. So to divide both rectangles in half, the line has to go through the centres of the two rectangles. Therefore there is exactly one line with the required property.

United Kingdom Mathematics Trust

Day Four

UKMT Lockdown Problem

1 2 3 4 5

The digits 1, 2, 3, 4, 5 may be used, once each, to make 120 different five-digit positive integers.

If these 120 positive integers are arranged in order, from smallest to largest, which is the 100th number in the list?

UKMT
United Kingdom Mathematics Trust

Day Five

UKMT Lockdown Solution

51342

The integers in the list with first digit 1 are followed by the digits 2, 3, 4, 5 in some order. These four digits may be arranged in order in $4 \times 3 \times 2 \times 1 = 24$ different ways. So there are 24 numbers in the list with first digit 1.
In a similar way there are 24 numbers in the list beginning with the digit 2, 24 beginning with the digit 3 and 24 beginning with the digit 4.
These numbers make up the first $4 \times 24 = 96$ integers in the list. These are followed by the integers that begin with the digit 5. Thus the list continues

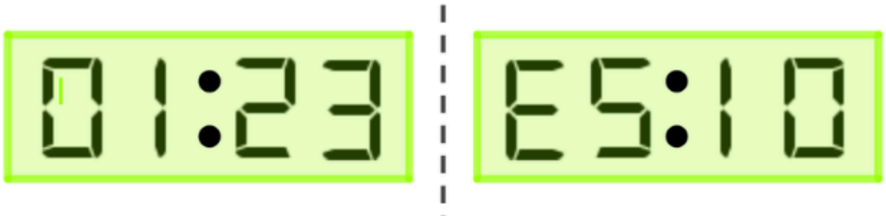
97. 51234
98. 51243
99. 51324
100. 51342,

and so we see that the 100th number in the list is 51342.

UKMT
United Kingdom Mathematics Trust

Day Five

UKMT Lockdown Problem



From the picture we see that when a 24-hour digital clock showing the time 01:23 is reflected in a vertical mirror the display does *not* look like an actual time.

How many times a day does a 24-hour digital clock show a time that also looks like a genuine time when it is reflected in a vertical mirror?
[Note that the clock will show 00:00 at midnight.]

UKMT
United Kingdom Mathematics Trust

Day Six

UKMT Lockdown Solution

121

The clock digits 0, 1 and 8 look the same after a vertical reflection. The digits 2 and 5 interchange. None of the other digits looks like a digit after a vertical reflection.
So the only times we need consider are those that use just the digits 0, 1, 2, 5 and 8.
The hour is given by a number from 0 to 24. The minutes are given by a number from 0 to 59.
It follows that there are just 11 hour numbers that reflect to a minute number, and vice versa. These are:
$00 \leftrightarrow 00;\ 01 \leftrightarrow 10;\ 02 \leftrightarrow 50;\ 05 \leftrightarrow 20;\ 10 \leftrightarrow 01;\ 11 \leftrightarrow 11;$
$12 \leftrightarrow 51;\ 15 \leftrightarrow 21;\ 20 \leftrightarrow 05;\ 21 \leftrightarrow 15$ and $22 \leftrightarrow 55$.
Because there are 11 possible values for the hour number and 11 possible values for the minutes number, there are $11 \times 11 = 121$ times that reflect to another time.

United Kingdom Mathematics Trust

Day Six

UKMT Lockdown Problem

79 992

a, b, c and d are four different digits.
They can be arranged in order in 24 ways, making 24 different four-digit numbers.

The sum of these 24 four-digit numbers is 79 992.

What is the smallest possible value of the product $a \times b \times c \times d$?

UKMT
United Kingdom Mathematics Trust

Day Seven

UKMT Lockdown Solution

36

In the 24 different four-digit numbers made up from the digits a, b, c and d each digit occurs 6 times in each of the thousands, hundreds, tens and ones columns.
Therefore the contribution of the occurrences of the digit a to the total is $6000a + 600a + 60a + 6a = 6666a$, and likewise for the digits b, c and d.
Therefore the total of the 24 four-digit numbers is
$6666a + 6666b + 6666c + 6666d = 6666(a + b + c + d)$.
It follows that $6666(a + b + c + d) = 79\,9992$ and hence
$a + b + c + d = 79\,992 \div 6666 = 12$.
There are just two ways to make a total of 12 from four different digits, $1 + 2 + 4 + 5$ and $1 + 2 + 3 + 6$.
$1 \times 2 \times 4 \times 5 = 40$ and $1 \times 2 \times 3 \times 6 = 36$. Hence the smallest possible product of the four digits is 36.

United Kingdom Mathematics Trust

Day Seven

UKMT Lockdown Problem

One day (before social distancing) Aaira and Birva were in the front half a long queue.

Aaira was ahead of Birva and there were nine people between them.

Aaria calculated the difference between the number of people behind her, and the number of people in front of her.

Birva did a similar calculation.

What was the difference between their two answers?

UKMT

United Kingdom Mathematics Trust

Day Eight

UKMT Lockdown Solution

20

← x people ↑ ← 9 people → ↑ y people →

Aaira Birva

Suppose that there are x people in front of Aaira, and y people behind Birva.
Then there are $9+1+y=y+10$ people behind Aaira, and $x+1+9=x+10$ people in front of Birva.
It follows that the number that Aaira calculates is $(y+10)-x$ and the number that Birva calculates is $y-(x+10)=y-x-10$.
Hence the difference between the numbers that they calculate is $(y+10-x)-(y-x-10)=20$.

United Kingdom Mathematics Trust

Day Eight

UKMT Lockdown Problem

On the first day after the flood, half of Noah's animals escaped.
On the second day, one third of the remainder wandered off.
On the third day, one quarter of the rest hopped it.

What fraction of Noah's original menagerie was then left?

Intermediate Mathematical Challenge, 1997

UKMT

United Kingdom Mathematics Trust

Day Nine

UKMT Lockdown Solution

$\mathbf{\frac{1}{4}}$

After half the animals escaped on the first day the fraction that was left was $\frac{1}{2}$.

One third of these wandered off on the second day. So the fraction of the original number of animals that wandered off was $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

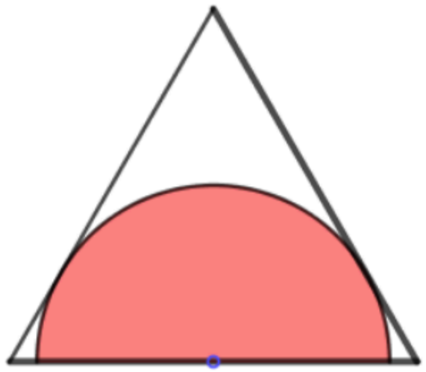
Therefore the fraction that remained after the second day was $\frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$.

One quarter of these hopped off on the third day. So the fraction of the original number that hopped off was $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

Therefore the fraction that remained after the third day was

$$\frac{1}{3} - \frac{1}{12} = \frac{4}{12} - \frac{1}{12} = \frac{3}{12} = \frac{1}{4}.$$

UKMT Lockdown Problem



The diagram shows a semi-circle inscribed in an an equilateral triangle.

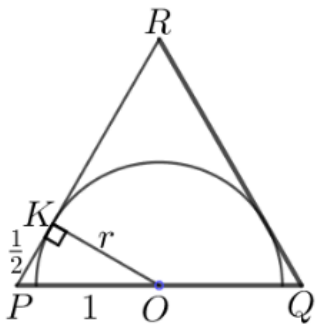
The sides of the triangle each have length 2 cm.

The centre of the semi-circle lies on one side of the triangle. The semi-circle touches the other two sides of the triangle.

What is the area of the semi-circle?

UKMT Lockdown Solution

$\boldsymbol{\frac{3}{8}\pi}$

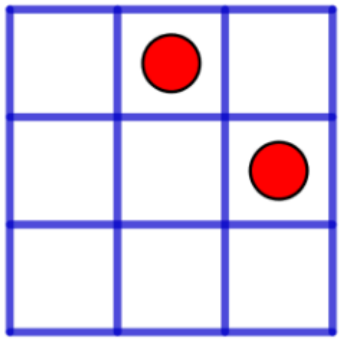


We label the points as shown and let r be the radius of the semicircle.
$\angle PKO$ is a right angle because the tangent PK to the semicircle is perpendicular to the radius OK.
Therefore the triangle PKO has angles 90°, 60° and 30°.
Thus it is half of an equilateral triangle. Hence $PK = \frac{1}{2}$.
Therefore, by Pythagoras' Theorem applied to the triangle PKO,
$\frac{1}{2}^2 + r^2 = 1^2$. Hence $r^2 = 1^2 - \frac{1}{2}^2 = 1 - \frac{1}{4} = \frac{3}{4}$.
The area of the semicircle $= \frac{1}{2}\pi r^2 = \frac{1}{2}(\pi \frac{3}{4}) = \frac{3}{8}\pi$.

UKMT
United Kingdom Mathematics Trust

Day Ten

UKMT Lockdown Problem



I wish to place two identical counters in the 3×3 grid so that they are not in the same row or columnn and they are not both on the same main diagonal. One way I could do this is shown above.

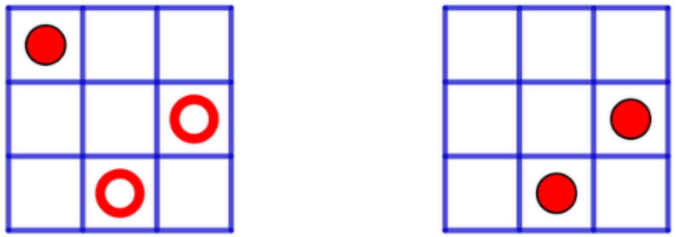
In how many different ways could I place the two counters?

United Kingdom Mathematics Trust

Day Eleven

UKMT Lockdown Solution

12



If one counter is in a corner square, there are just two possible positions for the second counter, as shown in the diagram on the left above.
Because there are 4 corners, there are 8 possible positions of this kind.
If neither counter is in a corner, the only possibility is that the two counters are at the midpoints of adjacent edges, as shown in the diagram on the right.
There are 4 possible positions of this type.
This makes a total of 12 positions.

UKMT Lockdown Problem

£ £ £

April, May and June have £9 between them.
April gave a half of her money to May. Next, May gave one third of what she then had to June. Finally, June gave a quarter of what she then had to April.

They all ended up with the same sum of money.

How much did each of them begin with?

UKMT
United Kingdom Mathematics Trust

Day Twelve

UKMT Lockdown Solution

April: £4, May: £2.50, June: £2.50.

It is easiest to work backwards from when they all ended up with the same amount of money.
Because they have £9 between them, this common amount is £3.
Then before June gave one quarter of her money to April, June will have had £$\frac{4}{3}(3)$ = £4. Then June gives £1 to April, who therefore before this had £3 − £1 = £2. Before May gave one third of her money to June, May will have had £$\frac{3}{2}(3)$ = £4.50. Thus May gave £1.50 to June. Therefore, before this June will have had £4 − £1.50 = £2.50.
To sum up so far, immediately after April has given one half of her money to May, the amounts that they have are;

April: £2, May: £4.50, June: £2.50.

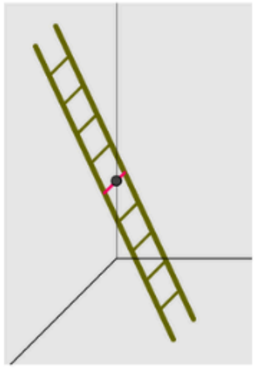
April gives one half of her money to May. Therefore before this April has £4. She gives £2 to May. Therefore, before this May had £4.50 − £2 = £2.50. Therefore the amounts they begin with are

April: £4, May: £2.50, June: £2.50.

UKMT
United Kingdom Mathematics Trust

Day Twelve

UKMT Lockdown Problem



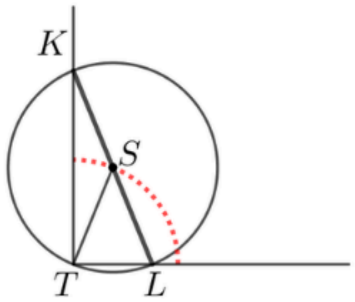
A ladder is leaning almost vertically against a wall.
It slides smoothly down the wall until it is lying flat on the floor.
A snail hangs on to the centre of the middle step as the ladder slides.
What is the shape of the path that the snail traces out as the ladder falls?
[As a follow-up question, you might like to consider the path traced out by a snail that is on a different step.]

UKMT
United Kingdom Mathematics Trust

Day Thirteen

UKMT Lockdown Solution

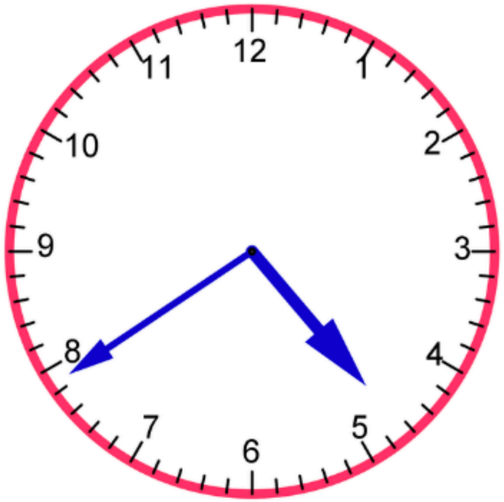
A quarter circle.



We take a vertical slice through the snail - without hurting the snail! The line segment KL represents the ladder, S marks the position of the snail. T is the point where the wall meets the floor.
Because $\angle KTL$ is a right angle, the circle with KL as diameter, and hence centre S, goes through T. Hence $TS = SL$. It follows that S lies on the circle with centre T and radius equal to half the length of the ladder. So, as the ladder slides S traces out the quarter circle shown.

If S is another point on the ladder, it traces out a quarter of an ellipse.

UKMT Lockdown Problem



How many times between midday and midnight is the hour hand of a clock at right angles to the minute hand?

Junior Mathematical Challenge, 1999

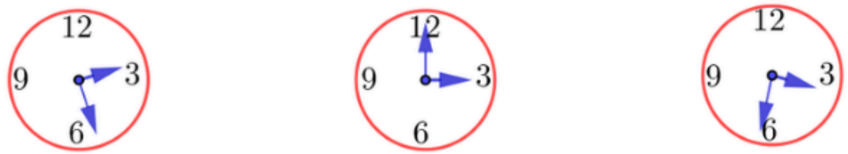
United Kingdom Mathematics Trust

Day Fourteen

UKMT Lockdown Solution

22

In general the hour hand and the minute hand are at right angles twice in each hour.
However, because the hands are at right angles at 3pm, they are at right angles just three times between 2pm and 4pm, at approximately 2.27pm, at 3pm and at approximately 3.32pm.



Similarly, because the hands are at right angles at 9pm, they are at right angles just three times between 8pm and 10pm.
It follows that the number of times between midday and midnight when the hands are at right angles is 22 $(= 12 \times 2 - 2)$.

UKMT Lockdown Problem

23	47	75

The integers from 1 to 100 are written on separate cards with one integer on each card. The cards are distributed between a group of people. Different people may receive different numbers of cards.

Each person counts the total number of 7s on the cards that they have been given.

It turns out that all these totals are different.

What is the largest possible number of people in the group?

UKMT

United Kingdom Mathematics Trust

Day Fifteen

UKMT Lockdown Solution

6

There are nine cards with a single 7 on them in the units (ones) place, and nine cards with a single 7 on them in the tens place. There are two 7s on the card on which 77 is written. The remaining 81 cards have no 7s on them. This makes a total of twenty 7s on the cards.

If there were seven or more people in the group holding cards with different numbers of 7s on them, the least number of 7s they could hold between them would be $0 + 1 + 2 + 3 + 4 + 5 + 6 = 21$. So there cannot be as many as seven people in the group.

However, with six people in the group the numbers of 7s that they hold could be 0, 2, 3, 4, 5 and 6, making a total of 20.

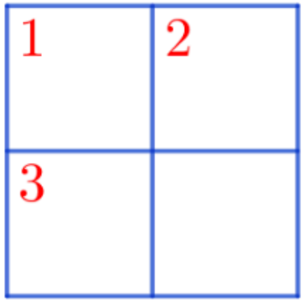
Hence the largest possible number of people in the group is 6.

United Kingdom Mathematics Trust

Day Fifteen

UKMT Lockdown Problem

Complete this crossnumber.



Across

1. A prime number
3. The square of a square

Down

1. The square of a square
2. A square

United Kingdom Mathematics Trust

Day Sixteen

UKMT Lockdown Solution

1 **8**	2 **3**
3 **1**	**6**

The only two-digit squares of squares are $(2^2)^2 = 4^2 = 16$ and $(3^2)^2 = 9^2 = 81$.

Therefore 1 Down is 81 and 3 Across is 16.

The two-digit squares with last digit 6 are 16 and 36.

If 2 Down were 16, 1 Across would be 81 which is not a prime. However if 2 Down is 36, 1 Across is 83 which is a prime. Hence 2 Down is 36.

Hence the solution is as above.

United Kingdom Mathematics Trust

Day Sixteen

UKMT Lockdown Problem

123 132 213 231 312 312

If the digits of a three digit number are all different they may be rearranged to make six different three-digit numbers. An example of this is shown above.
If two of the digits of a three-digit number are the same, its digits may be rearranged to make three different three-digit numbers.

Find all the three-digit prime numbers that have the property that each rearrangement of its digits produces another prime number.

United Kingdom Mathematics Trust

Day Seventeen

UKMT Lockdown Solution

$113, 131, 311 \quad 337, 373, 733 \quad 199, 919, 991$

If the units digit (sometimes known as the *ones digit*) of a three-digit number is even, the number is even and so it not prime. If it is 5, the number is divisible by 5 and so is not prime. Therefore we need only consider three-digit numbers which only use the digits 1, 3, 7 and 9.

If the digits add up to a multiple of 3, the number is a multiple of three and so is not prime. This rules out 111, 117, 177, 333, 339, 399, 777 and 999, and any rearrangment of these.

We also have $119 = 7 \times 17$, $133 = 7 \times 19$, $319 = 11 \times 29$, $371 = 7 \times 53$, $737 = 11 \times 67$, $779 = 19 \times 41$, $791 = 7 \times 113$, $793 = 13 \times 61$ and $979 = 11 \times 89$.

This rules out all the other possible three-digit numbers other than those shown in the answer above.

You may check that these are all prime numbers.

UKMT Lockdown Problem

$\times 7$ and $\times 21$

(a) First, find all the two-digit integers that are equal to 7 times the sum of their digits.

(b) Then, find all the three-digit integers that are equal to 21 times the sum of their digits.

United Kingdom Mathematics Trust

Day Eighteen

UKMT Lockdown Solution

(a) 21, 42, 63, 84. (b) 378.

(a) The two-digit number "ab" represents the integer $10a + b$. Therefore we seek solutions of the equation $10a + b = 7(a + b)$. It follows that $3a = 6b$ and hence $a = 2b$. Because a and b are digits, and cannot both be 0, the only solutions are $a = 2, b = 1$; $a = 4, b = 2$; $a = 6, b = 3$ and $a = 8, b = 4$. Therefore the required two-digit integers are 21, 42, 63 and 84.

(b) The three-digit number "abc" represents the number $100a + 10b + c$. Therefore we need to solve the equation $100a + 10b + c = 21(a + b + c)$. This equation may be rearranged as $79a = 11b + 20c$. Now $0 \leq b, c \leq 9$, and so $79a = 11b + 10c \leq 99 + 180 = 279$. Hence $a \leq 3$. When $a = 1$, we obtain $11b + 20c = 79$, an equation with no positive integer solutions. When $a = 2$, we obtain $11b + 20c = 158$ which also has no positive integer solutions. When $x = 3$, we have $11b + 20c = 237$. It may be checked that the only positive integer solution is $b = 7, c = 8$. Therefore 378 is the only three-digit integer which is equal to 21 times the sum of its digits.

UKMT

United Kingdom Mathematics Trust

Day Eighteen

UKMT Lockdown Problem

223506

The six-digit integer shown above has an even number of even digits, namely the two 2s, the 0 and the 6, making four even digits in total.

How many six-digit integers have an even number of even digits?

United Kingdom Mathematics Trust

Day Twenty

UKMT Lockdown Solution

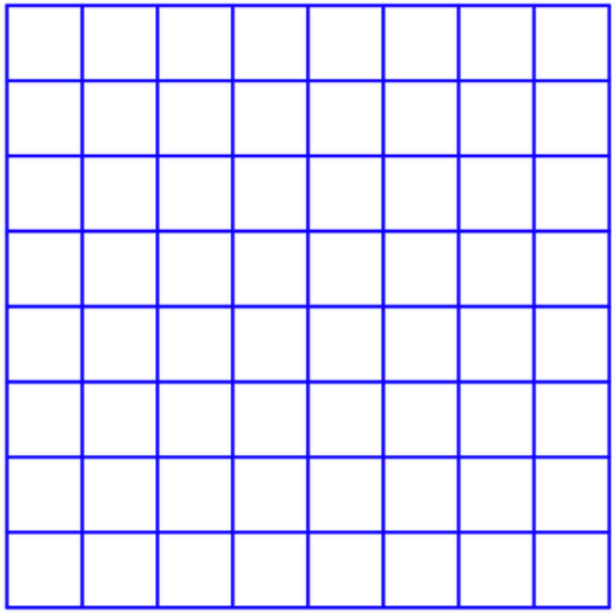
450 000

There are 900 000 six-digit integers, from 100 000 to 999 999. We can arrange these into 450 000 pairs of consecutive integers: (100 000,100 001), (100 002,100 003), ... ,(999 998,999 999). The first five digits of the integers in each pair are the same. Also, the last digit of the first integer in each pair is even, and the last digit of the second integer is odd. It follows that exactly one integer in each pair has an even number of even digits. Therefore there are 450 000 six-digit integers with an even number of even digits.

United Kingdom Mathematics Trust

Day Twenty

UKMT Lockdown Problem



How many squares of any size are there in this 8×8 grid?

United Kingdom Mathematics Trust

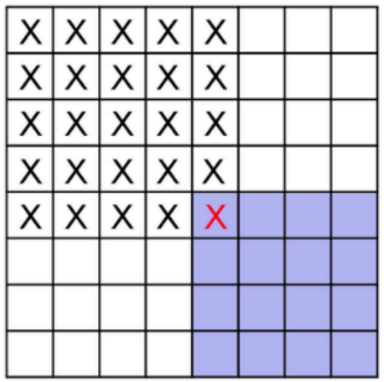
Day Twenty One

UKMT Lockdown Solution

204

There are 8^2=64 squares of size 1×1 in the grid.
When it comes to counting larger squares it helps to be systematic.
We choose to count them by counting the number of 1×1 squares that can occur as the top left-hand corner of the larger squares.
The top left-hand corner of a 2×2 square is in one of first 7 rows and first 7 columns. Therefore there are $7^2 = 49$ squares of size 2×2 in the grid.

This pattern continues. For example, the diagram shows that the top left-hand corner of a 4×4 square can be in any of the $5^2 = 25$ squares marked with a $\times$.
It follows that the total number of squares in the grid of any size is
$8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 204$.



UKMT
United Kingdom Mathematics Trust

Day Twenty One

UKMT Lockdown Problem

Alice's age is the sum of the ages of Bob and Charlotte.

Five years ago Alice was twice as old as Bob was then.

In how many years time will Alice be twice as old as Charlotte will then be?

United Kingdom Mathematics Trust

Day Twenty Two

UKMT Lockdown Solution

5 years

We let the ages in years of Alice, Bob and Charlotte be a, b and c, respectively. Because Alice's age is the combined age of Bob and Charlotte, $a = b + c$. (1)

Five years ago the ages of Alice and Bob were $a - 5$ and $b - 5$, and Alice was twice the age of Bob. Hence $a - 5 = 2(b - 5)$. Hence $a + 5 = 2b$ and so $b = \frac{1}{2}a + \frac{5}{2}$.

Therefore, by (1), $a = (\frac{1}{2}a + \frac{5}{2}) + c$. Hence $a = 2c + 5$. (2)

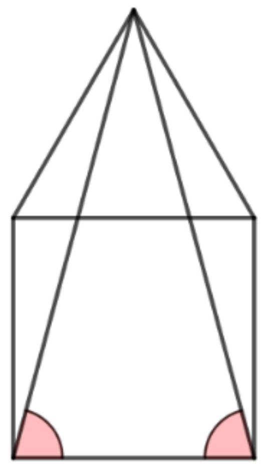
Now suppose that Alice is twice Charlotte's age in k year's time. Then $a + k = 2(c + k)$, that is, $a = 2c + k$. It follows from (2) that $2c + k = 2c + 5$. Hence $k = 5$.

Therefore Alice is twice Charlotte's age in five year's time.

United Kingdom Mathematics Trust

Day Twenty Two

UKMT Lockdown Problem

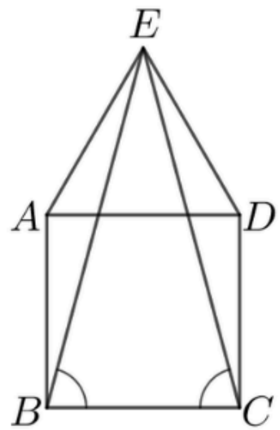


The diagram shows a two-dimensional shape.
It is made up of a square and an equilateral triangle that have an edge in common.
Two other lines have been added, as shown.

What is the size, in degrees, of the marked angles?

UKMT Lockdown Solution

75°



Let the points be labelled as shown.
Because $ABCD$ is a square, and ADE is an equilateral triangle, we have $AB = AD = AE$.
Therefore the triangle ABE is isosceles.
Hence $\angle ABE = \angle AEB$.

$\angle BAE = \angle BAD + \angle DAE = 90^\circ + 60^\circ = 150^\circ$. Therefore, because the sum of the angles in a triangle is 180°, we deduce that $\angle ABE = \angle AED = \frac{1}{2}(180^\circ - 150^\circ) = \frac{1}{2}(30^\circ) = 15^\circ$.

Hence $\angle EBC = \angle ABC - \angle ABE = 90^\circ - 15^\circ = 75^\circ$.
Similarly, $\angle ECB = 75^\circ$.
Exercise: Use this diagram to find $\tan 75^\circ$ in terms of surds.

UKMT
United Kingdom Mathematics Trust

Day Twenty Three

UKMT Lockdown Problem

The diagram shows six small squares made from matches.

What is the smallest number of matches you need to remove to leave just three small squares?

Junior Mathematical Challenge, 2001

United Kingdom Mathematics Trust

Day Twenty Four

UKMT Lockdown Solution

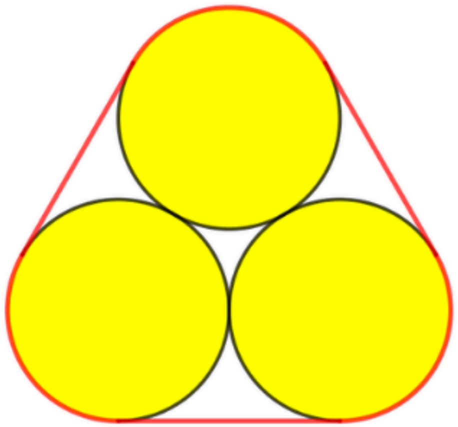
5

The diagram shows how five matches may be removed to leave three small squares.
Three small squares need 12 matches if they have no edges in common, and fewer matches if they share edges. There are 17 matches to start with.
Hence $17 - 12 = 5$ is the smallest number of matches that need to be removed.

United Kingdom Mathematics Trust

Day Twenty Four

UKMT Lockdown Problem



The perimeter of the figure shown in the diagram consists of parts of the circumferences of the three circles and parts of lines that are tangents to two of the circles, as shown.
Each of the circles has radius 1, and touches the other two circles.

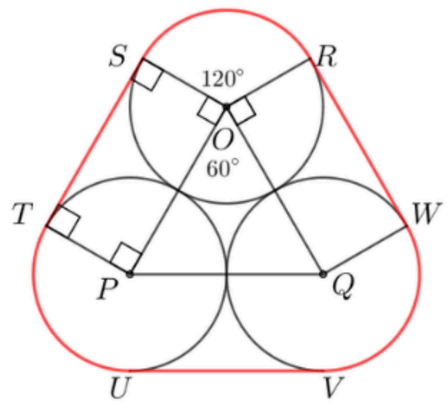
What is the length of the perimeter of the figure?

UKMT
United Kingdom Mathematics Trust

Day Twenty Five

UKMT Lockdown Solution

$6 + 2\pi$



We label the points as shown. The angles OST and STP are right angles since the radii OS and PT are perpendicular to tangent ST. Hence $OPTS$ is a rectangle and $ST = OP = 2$. Similarly, $UV = RW = 2$. Hence the total length of these three line segments is $2 + 2 + 2 = 6$. The triangle OPQ is equilateral. Therefore, from the angles at the point O we deduce that $\angle ROS = 360^\circ - 90^\circ - 60^\circ - 90^\circ = 120^\circ$. Therefore, the arc RS is one third of the circumference of the circle, and likewise for the arcs TU and VW. Therefore the total length of these three arcs is equal to the circumference of a circle of radius 1, and hence is 2π. Therefore the total length of the perimeter is $6 + 2\pi$.

United Kingdom Mathematics Trust

Day Twenty Five

UKMT Lockdown Problem

$$\begin{array}{r} 2\,0\,2\,1 \\ +\quad J\,A\,N \\ \hline G\,0\,0\,D \\ \hline \end{array} \qquad \begin{array}{r} 2\,0\,2\,1 \\ +\quad D\,E\,C \\ \hline Y\,E\,A\,R \\ \hline \end{array} \qquad \begin{array}{r} J\,A\,N \\ +\quad D\,E\,C \\ \hline B\,E\,S\,T \\ \hline \end{array}$$

In these addition sums each letter represents a non-zero digit.

Each non-zero digit is represented by at least one of the letters.

Some digits are represented by more than one letter.

[In “G00D”, the “0”s are the digit “zero”, not a letter.]

Which number is represented by GREAT?

United Kingdom Mathematics Trust

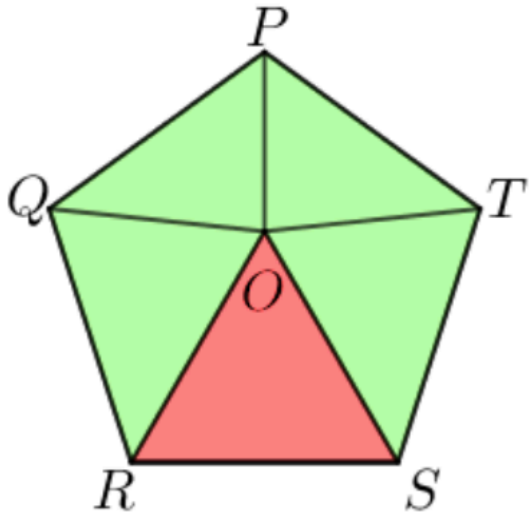
Day Twenty Six

UKMT Lockdown Solution

33687

From the sum on the left, because $D \neq 0$, $D = N+1$, $A = 8$, $J = 9$ and $G = 3$.
Similarly, from the middle sum, $C \neq 0$, and hence $R = C+1$, $E = 6$, $D = 6$ and $Y = 2$. Therefore, as $D = N+1$, $N = 5$.
This leaves B, C, R, S and T for which we have not yet found values, and the digits 1, 4 and 7 for which we have not yet found corresponding letters.
It may then be seen from the sum on the right, that $B = 1$, $C = 2$, $R = 3$, $S = 4$ and $T = 7$.
It follows that GREAT corresponds to 33687.

UKMT Lockdown Problem



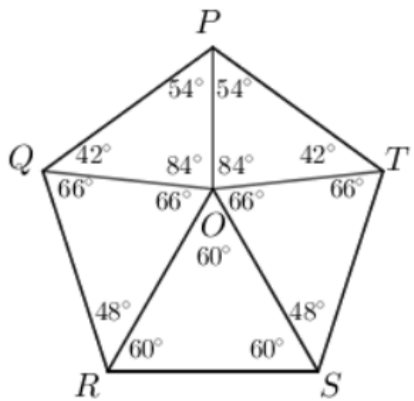
$PQRST$ is a regular pentagon and ORS is an equilateral triangle.

Find the sizes of all the angles in all the triangles in the diagram.

United Kingdom Mathematics Trust

Day Twenty Seven

UKMT Lockdown Solution

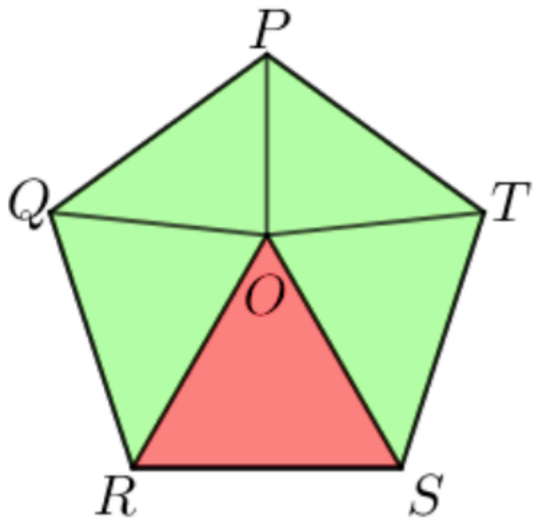


ORS is an equilateral triangle and therefore $\angle ORS = \angle RSO = \angle SOR = 60°$. Each interior angle of a regular pentagon is $108°$. Hence $\angle ORQ = \angle OST = 108° - 60° = 48°$. Since $OR = RS = RQ$, the triangle ORQ is isosceles. Hence $\angle RQO = \angle ROQ$. Therefore, as the sum of the angles in the triangle ROQ is $180°, \angle RQO = \angle ROQ = \frac{1}{2}(180° - 48°) = 66°$. Similarly $\angle SOT = \angle OTS = 66°$. $\angle PQO = \angle PTO = 108° - 66° = 42°$. By the symmetry of the figure, $\angle QPO = \angle TPO = \frac{1}{2}(108°) = 54°$.
Hence $\angle POQ = \angle POT = 180° - (42° + 54°) = 84°$.

United Kingdom Mathematics Trust

Day Twenty Seven

UKMT Lockdown Problem



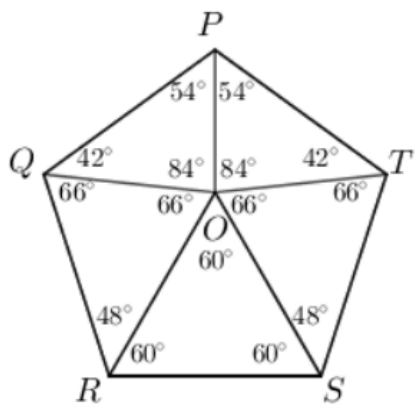
$PQRST$ is a regular pentagon and ORS is an equilateral triangle.

Find the sizes of all the angles in all the triangles in the diagram.

United Kingdom Mathematics Trust

Day Twenty Seven

UKMT Lockdown Solution



ORS is an equilateral triangle and therefore $\angle ORS = \angle RSO = \angle SOR = 60^\circ$. Each interior angle of a regular pentagon is 108°. Hence $\angle ORQ = \angle OST = 108^\circ - 60^\circ = 48^\circ$. Since $OR = RS = RQ$, the triangle ORQ is isosceles. Hence $\angle RQO = \angle ROQ$. Therefore, as the sum of the angles in the triangle ROQ is $180^\circ, \angle RQO = \angle ROQ = \frac{1}{2}(180^\circ - 48^\circ) = 66^\circ$. Similarly $\angle SOT = \angle OTS = 66^\circ$. $\angle PQO = \angle PTO = 108^\circ - 66^\circ = 42^\circ$. By the symmetry of the figure, $\angle QPO = \angle TPO = \frac{1}{2}(108^\circ) = 54^\circ$.
Hence $\angle POQ = \angle POT = 180^\circ - (42^\circ + 54^\circ) = 84^\circ$.

UKMT

United Kingdom Mathematics Trust

Day Twenty Seven

UKMT Lockdown Problem

1

3 5

7 9 11

.. ..

.. ..

The odd numbers are arranged in a triangle, as shown above. [Only the numbers in the first three rows are shown.]

What is the sum of the numbers in the 20th row of this triangle?

United Kingdom Mathematics Trust

Day Twenty Eight

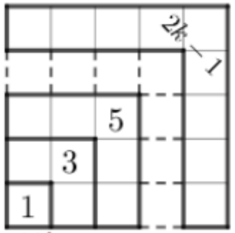
UKMT Lockdown Solution

8000

You will see that the sums of the numbers in the first three rows of this triangle are as follows: row 1: $1 = 1^3$;

row 2: $3 + 5 = 8 = 2^3$; row 3: $7 + 9 + 11 = 27 = 3^3$.

It may be proved (see below) that this pattern continues. Therefore the sum of the numbers in the 20th row is $20^3 = 8000$.

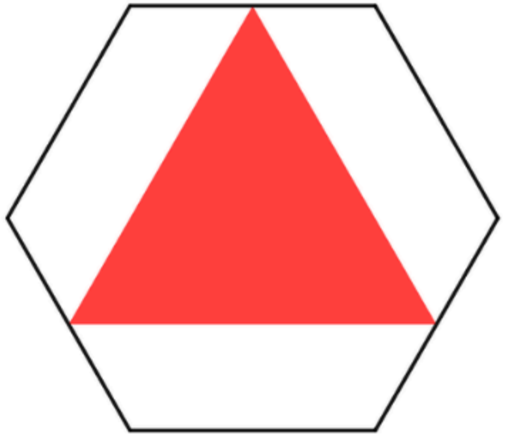


[To prove the general result you first need to show that the sum of the first k odd numbers is k^2. This may be done using algebra, or by geometry, as in the diagram alongside. Next, note that the total number of numbers in the first n rows of the triangle is $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$. Hence, putting $k = \frac{1}{2}n(n+1)$, we have that the sum of the numbers in the first n rows of the triangle is $\left(\frac{1}{2}n(n+1)\right)^2$. Similarly, the sum of the numbers in the first $n-1$ rows of the triangle is $\left(\frac{1}{2}(n-1)n\right)^2$. Hence the sum of the numbers in the nth row is $\left(\frac{1}{2}n(n+1)\right)^2 - \left(\frac{1}{2}(n-1)n\right)^2$. Check that this simplifies to n^3.]

United Kingdom Mathematics Trust

Day Twenty Eight

UKMT Lockdown Problem



The diagram shows an equilateral triangle with its corners at the mid-points of alternate sides of a regular hexagon.

What is the area of the triangle as a fraction of the area of the hexagon?

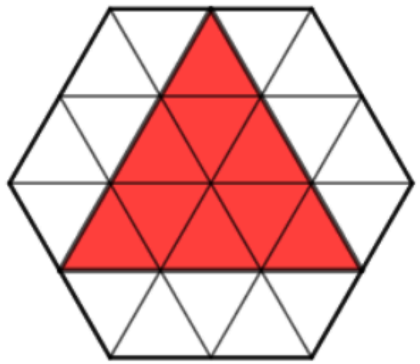
Junior Mathematical Challenge, 2006.

United Kingdom Mathematics Trust

Day Twenty Nine

UKMT Lockdown Solution

$\mathbf{\frac{3}{8}}$

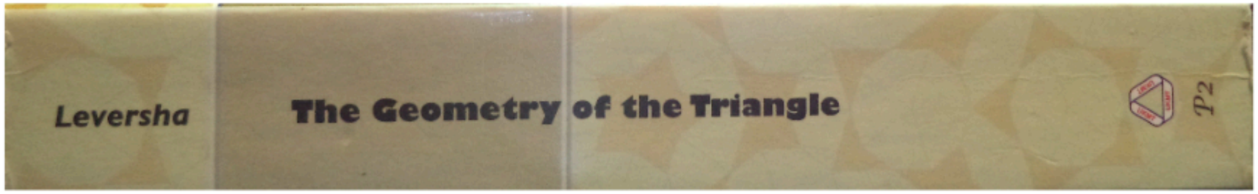


We divide the hexagon into 24 small equilateral triangles all of the same size. The large equilateral triangle is made up of 9 of these small triangles. Therefore the area of this triangle as a fraction of the area of the hexagon is $\frac{9}{24} = \frac{3}{8}$.

United Kingdom Mathematics Trust

Day Twenty Nine

UKMT Lockdown Problem



The pages of a book are numbered 1, 2, 3, 4, and so on. The digit 1 is used 189 times in numbering the pages.

How many pages are there in the book?

United Kingdom Mathematics Trust

Day Thirty

UKMT Lockdown Solution

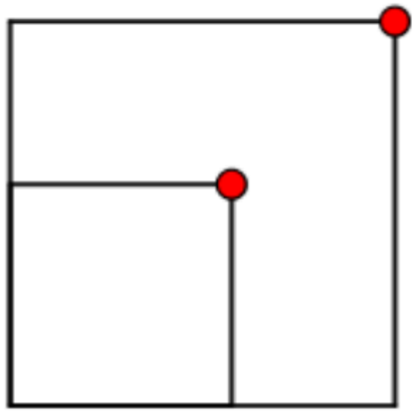
416

In the page numbers from 1 to 99, the digit 1 occurs 10 times as the units (ones) digit in the page numbers, 1, 11, 21, ..., 91, and 10 times as the tens digit in the page numbers 10, 11, 12, ..., 19. This makes a total of 20. In the page numbers 100, 101, .., 199 the digit 1 again occurs 20 times in the units and tens places, but also 100 times in the hundreds place.
So the digit 1 occurs $20 + 20 + 100 = 140$ times in the first 199 page numbers. It occurs 20 times in the page numbers from 200 to 299 and 20 times in the page numbers from 300 to 399. Hence it occurs $140 + 20 + 20 = 180$ times in the first 399 pages. The next 9 occurrences are in the page numbers 401, 410, 411, 412, 413, 414, 415 and 416. So there 189 occurrences in pages 1 to 416, and hence 190 occurences in pages 1 to 417. So the book has 416 pages.

United Kingdom Mathematics Trust

Day Thirty

UKMT Lockdown Problem



The diagram shows two squares.

The smaller square has area 18 cm^2.

The larger square has area 50 cm^2.

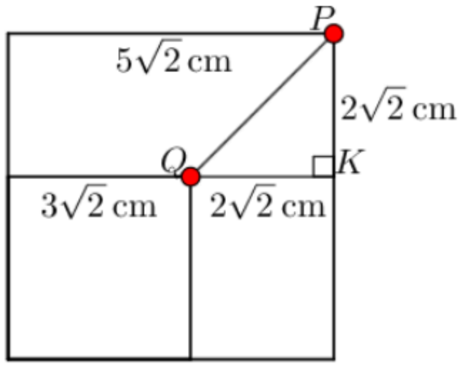
What is the distance between the two marked vertices?

United Kingdom Mathematics Trust

Day Thirty One

UKMT Lockdown Solution

4 cm



Let P, Q and K be the points shown. The smaller square has area 18 cm^2. Hence the length of its sides is $\sqrt{18}$ cm, that is $3\sqrt{2}$ cm. The larger square has area 50 cm^2, and hence the length of its sides is $\sqrt{50}$ cm, that is $5\sqrt{2}$ cm. It follows that $QK = (5\sqrt{2} - 3\sqrt{2})$ cm $= 2\sqrt{2}$ cm. Similarly, $KP = 2\sqrt{2}$ cm. By Pythagoras' Theorem applied to the triangle QKP, we have $QP^2 = QK^2 + KP^2$. Hence the length of QP, in cm, is

$$\sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = \sqrt{16} = 4.$$

United Kingdom Mathematics Trust

Day Thirty One

UKMT Lockdown Problem

$$10^{16} - 3^{16}$$

Which is the smallest prime number that that is a factor of $10^{16} - 3^{16}$?

United Kingdom Mathematics Trust

Day Thirty Two

UKMT Lockdown Solution

7

To show that 7 is the smallest prime factor of $10^{16} - 3^{16}$ we need to show that 2, 3 and 5 are not factors, but 7 is. There are several ways to do this. Here is one:

10^{16} is divisible by 2 and 5, but 3^{16} is divisible by neither. Hence neither 2 nor 5 is a factor of $10^{16} - 3^{16}$. Also,

$10^{16} - 3^{16} = (999\ldots999 + 1) - 3^{16} = 3(333\ldots333 - 3^{15}) + 1$,

and hence is not divisible by 3.

To show that 7 a factor of $10^{16} - 3^{16}$ we use the "difference of two squares" factorization, $a^2 - b^2 = (a-b)(a+b)$. This gives $x^{16} - y^{16} = (x^8 - y^8)(x^8 + y^8) = (x^4 - y^4)(x^4 + y^4)(x^8 + y^8)$

$= (x^2 - y^2)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8)$

$= (x - y)(x + y)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8)$. So $x - y$ is a factor of $x^{16} - y^{16}$. Now put $x = 10$ and $y = 3$.

United Kingdom Mathematics Trust

Day Thirty Two

UKMT Lockdown Problem

Apples, bananas and oranges cost 10 pence each.
I have £1 to spend on fruit. I decide to spend it all.

How many different combinations of fruit can I buy?

United Kingdom Mathematics Trust

Day Thirty Three

UKMT Lockdown Solution

66

We wish to count the number of ways of splitting up a group of 10 fruit into apples, bananas and oranges.
We give one method for doing this here.
We count the choices according to the number of apples.
[Note that once we have decided the numbers of apples and bananas, the number of oranges is
10 − the number of apples − the number of bananas.]

10 apples: 0 bananas - 1 combination.
9 apples: 0 or 1 bananas - 2 combinations.
8 apples: 0, 1 or 2 bananas - 3 combinations
⋮
0 apples: 0, 1, 2, ... , 10 bananas - 11 combinations.

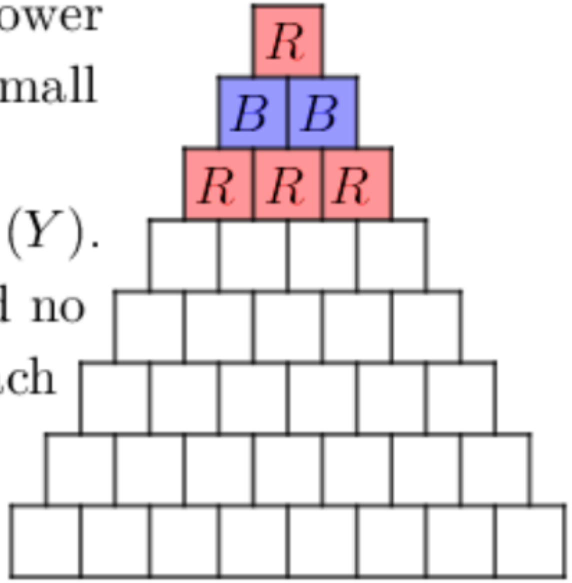
We see that the total number of possible combinations is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = 66$.

UKMT
United Kingdom Mathematics Trust

Day Thirty Three

UKMT Lockdown Problem

Donna is making a coloured tower as shown. She has thirty-six small cubes, with equal numbers of red (R), blue (B) and yellow (Y). Each row is of one colour, and no two rows which are next to each other are the same colour. The top three rows are coloured as shown.



What colour must the bottom row be?

Junior Mathematical Challenge, 1997.

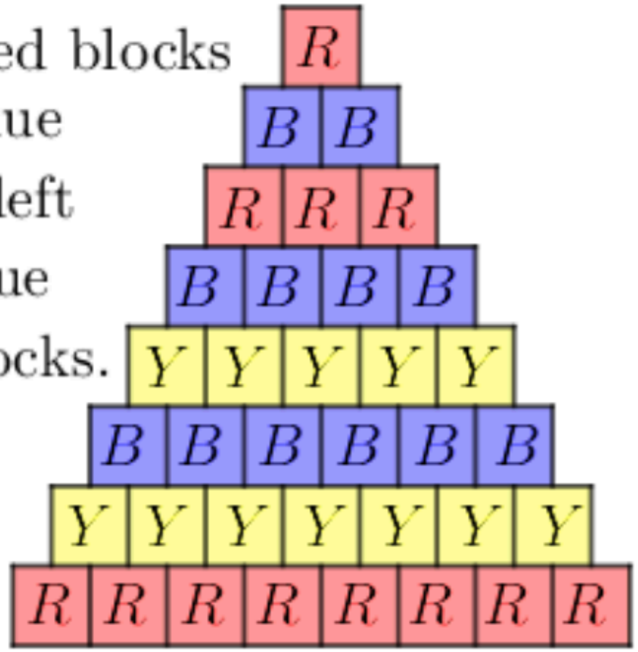
United Kingdom Mathematics Trust

Day Thirty Four

UKMT Lockdown Solution

red

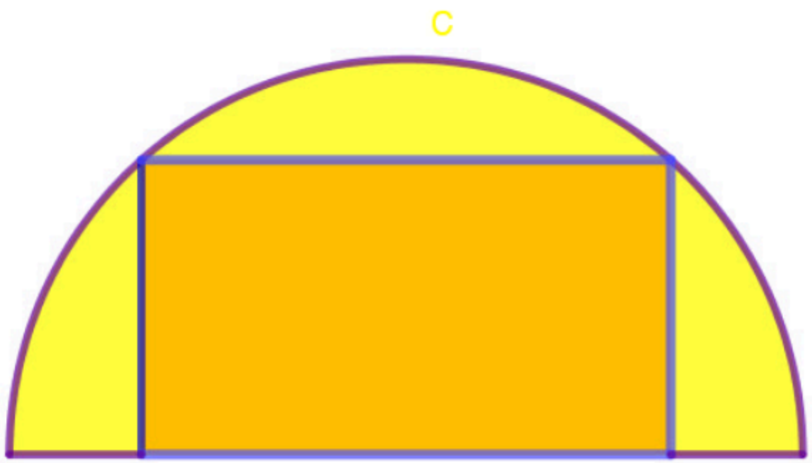
After Donna has used red blocks for rows 1 and 3, and blue blocks for row 2, she is left with 8 red blocks, 10 blue blocks and 12 yellow blocks. The only way these can be fitted into complete rows as is shown.



United Kingdom Mathematics Trust

Day Thirty Four

UKMT Lockdown Problem



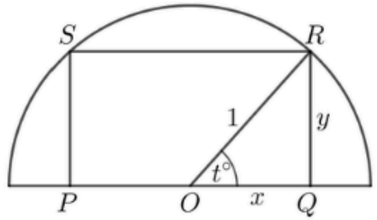
The semicircle shown has radius 1.

What is the largest possible area of a rectangle with two vertices on the semicircle and two on the diameter of the semicircle?

Day Thirty Five

UKMT Lockdown Solution

1



Let O, P, Q, R and S be the points shown in the diagram. Let x, y be the lengths of OQ and QR, respectively. Then OP also has length x. So PQ has length $2x$.

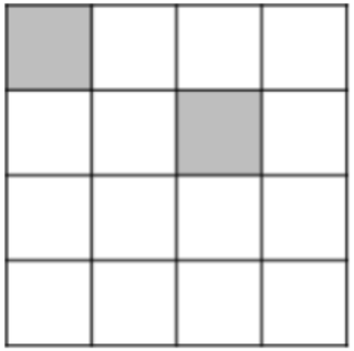
By Pythagoras' theorem applied to the triangle OQR, $x^2 + y^2 = 1$ and therefore $y^2 = 1 - x^2$. The area of the rectangle $PQRS = 2xy = 2\sqrt{x^2y^2} = 2\sqrt{x^2(1-x^2)} = 2\sqrt{x^2 - x^4} = 2\sqrt{\frac{1}{4} - (\frac{1}{2} - x^2)^2}$. Hence the area has a maximum value of $2\sqrt{\frac{1}{4}} = 2 \times \frac{1}{2} = 1$, when $\frac{1}{2} - x^2 = 0$.

Alternative method: Let $\angle ROQ = t^\circ$. Then $OQ = \cos t^\circ$ and $RQ = \sin t^\circ$. Similarly, $OP = \cos t^\circ$. Hence $PQ = 2\cos t^\circ$. Therefore the area of the rectangle $PQRS$ is $PQ \times RS = 2\cos t^\circ \sin t^\circ = \sin 2t^\circ$. Therefore the maximum value of the area is 1 when $2t = 90^\circ$ and hence $t = 45^\circ$.

United Kingdom Mathematics Trust

Day Thirty Five

UKMT Lockdown Problem



The diagram shows a 4×4 grid in which two cells that do not share an edge are shaded.

(a) How many different pairs of cells that do not share an edge are there in the 4×4 grid?

(b) How many different pairs of cells that do not share an edge are there in an 8×8 grid?

(c) Find a formula for the number of pairs of cells that do not share an edge in an $n \times n$ grid.

UKMT
United Kingdom Mathematics Trust

Day Thirty Six

UKMT Lockdown Solution

(a) 96; (b) 1904; (c) $\frac{1}{2}\,n(n-1)(n^2+n-4)$

We first count the total number of pairs of cells in the grid. Then we we subtract the number of pairs of cells that share an edge.

(a) There are 16 cells in the 4×4 grid. So the first cell of a pair may be chosen in 16 ways, and then the second in 15 ways. Because the order does not matter, this counts each pair of cells twice. So there are $\frac{1}{2}(16 \times 15) = 120$ different pairs of cells.

2	3	3	2
3	4	4	3
3	4	4	3
2	3	3	2

In the diagram each cell contains the number of cells that it shares an edge with. The total of these numbers is $4 \times 2 + 8 \times 3 + 4 \times 4 = 48$. This counts each pair of cells that share an edge twice. So there are $\frac{1}{2}(48) = 24$ such pairs. Hence there are $120 - 24 = 96$ pairs of cells that don't share an edge.

(b) The same method shows that in the 8×8 grid the number of such pairs is $\frac{1}{2}\big((64 \times 63) - (4 \times 2 + 24 \times 3 + 36 \times 4)\big) = 1904$.

(c) Similarly, for an $n \times n$ grid the number of such pairs of cells is

$$\frac{1}{2}\big((n^2 \times (n^2 - 1)) - ((4 \times 2) + (4(n-2) \times 3) + ((n-2)^2 \times 4))\big)$$

$$= \frac{1}{2}(n^4 - 5n^2 + 4n) = \frac{1}{2}n(n-1)(n^2 + n - 4).$$

United Kingdom Mathematics Trust

Day Thirty Six

UKMT Lockdown Problem

On July 23rd Sam the Scottish snail sets out for his home in Edinburgh for Lammas (1st August). His home is 5 cms short of 32 metres away. Since he is old, each day Sam can only travel half the distance of the previous day. The first day he manages just 16 metres.

Does he get home in time? If so, on which day?

United Kingdom Mathematics Trust

Day Thirty Seven

UKMT Lockdown Solution

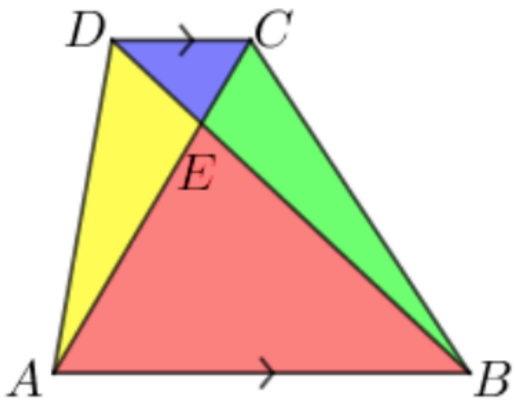
Yes. Sam the snail reaches home on August 1st.

Sam wants to end up 31.95 metres from his starting point. The number of days he takes to do this (if he can) is the least number of terms in the sum $16\,\mathrm{m} + 8\,\mathrm{m} + 4\,\mathrm{m} + \ldots$ that takes him past $31.95\,\mathrm{m}$. But, rather than add these term by term, it is a bit easier to find out *how far short* of $32\,\mathrm{m}$ he is after each day. We readily see that, after days $1, 2, 3$,... Sam is $32/2$, $32/4$, $32/8$, ... metres short of 32 metres, respectively. Now 5cms is $1/640$th of 32 metres. So Sam will arrive home on day n, where n is the smallest positive integer for which $1/2^n < 1/640$. This means that $n = 10$, so that Sam arrives at his home on August 1st.

UKMT United Kingdom Mathematics Trust

Day Thirty Seven

UKMT Lockdown Problem



$ABCD$ is a trapezium with AB parallel to DC.

The diagonals AC and BD meet at the point E.

The ratio
area of triangle AEB : area of triangle $DEC = 9 : 1$.

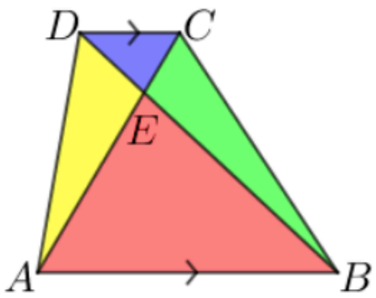
What is the ratio
area of triangle AEB : area of triangle AED ?

United Kingdom Mathematics Trust

Day Thirty Eight

UKMT Lockdown Solution

3 : 1



Because AB is parallel to DC, we have $\angle EAB = \angle ECD$ and $\angle EBA = \angle EDC$. Therefore the triangle AEB is similar to the triangle CED.

The ratio of the areas of similar triangles is equal to the ratio of the squares of the lengths of corresponding sides. Therefore, since area $\Delta AEB : \Delta DEC = 9 : 1$, it follows that $BE : ED = 3 : 1$.

The triangles AEB and AED have the same height when considered as each having a base on the line BD.

Hence area $\Delta AEB : \Delta AED = BE : ED = 3 : 1$.

UKMT Lockdown Problem

13		
5		15
x		

In a magic square, each row, each column and both main diagonals have the same total.

What number should replace x in this partially completed magic square?

Junior Mathematical Challenge, 2000.

UKMT
United Kingdom Mathematics Trust

Day Thirty Nine

UKMT Lockdown Solution

12

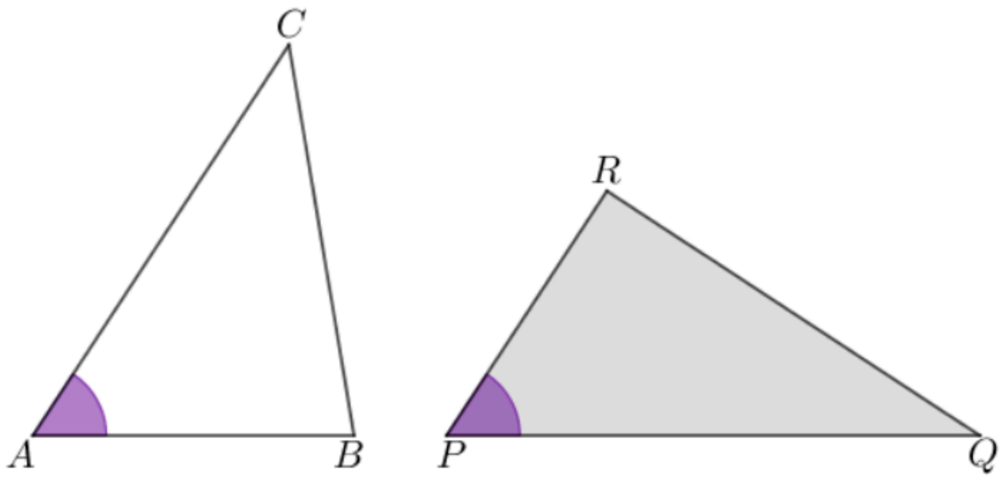
13		c $x-4$
5	a $x-2$	15
x		b 7

Let a, b and c be the numbers in the central, top-right and bottom-right squares, respectively.
The first column and middle row totals are equal.
Therefore $13+5+x=5+a+15$. Hence $a=x-2$.
Then, from the first column and top-left to bottom-right diagonal, $13+5+x=13+(x-2)+b$. Hence $b=7$.
Next, from the first and third columns we have $13+5+x=c+15+7$. Hence $c=x-4$.
Finally from the first column and the top-right to bottom-left diagonal, $13+5+x=(x-4)+(x-2)+x$.
Hence $2x-6=18$. So $2x=24$. Therefore $x=12$.

United Kingdom Mathematics Trust

Day Thirty Nine

UKMT Lockdown Problem

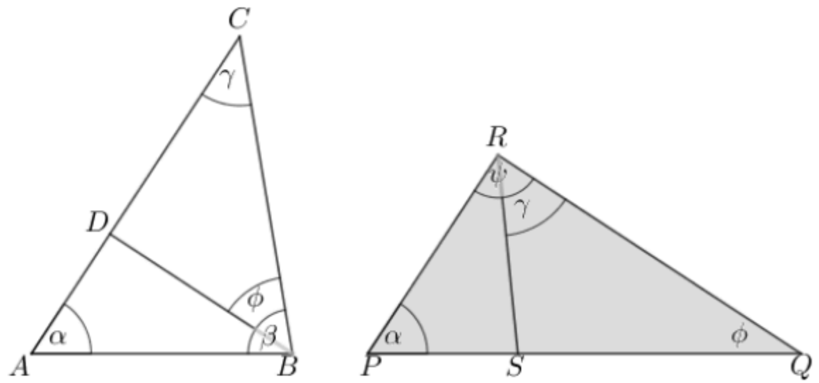


The unshaded triangle ABC and the shaded triangle PQR are *not* similar, but $\angle CAB = \angle RPQ$.
Assume that $\angle CAB \neq 90^\circ$.
Show how each triangle can be divided into two triangles by a straight line cut, so as to form two pairs of *similar* triangles, each pair consisting of one unshaded triangle and one shaded triangle.

United Kingdom Mathematics Trust

Day Forty

UKMT Lockdown Solution



Let $\angle CAB = \angle RPQ = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$, $\angle PQR = \phi$ and $\angle QRP = \psi$. Because the sum of the angles in a triangle is 180°, we have $\alpha + \beta + \gamma = \alpha + \phi + \psi$. Hence $\beta + \gamma = \phi + \psi$. (1) Because the triangles are not similar, $\gamma \neq \psi$. We suppose first that $\gamma < \psi$. Then, by (1), $\beta > \phi$. It follows that we can find D on AC so that $\angle DBC = \phi$, and S on PQ so that $\angle SRQ = \gamma$. Since $\angle DBC = \angle SQR$ and $\angle BCD = \angle QRS$, the triangles BCD and QRS are similar. By (1), $\beta - \phi = \psi - \gamma$. Hence $\angle ABD = \angle SRP$. We also have $\angle DAB = \angle RPS$. Therefore the triangles DAB and SPR are similar. Thus BD and RS are the required straight line cuts. If $\gamma > \psi$, then, by (1), $\beta < \phi$ and this case is similar.

UKMT
United Kingdom Mathematics Trust

Day Forty